Stochastic Time

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What is time?

- Time is an elastic one-dimensional manifold.
- It is used to order recognizable events generated by systems.
- When the events are generated by physical oscillators accepted by large communities, we speak of *clock time*.
- Lots of systems produce events, but are their clock times always interesting?
- Time is the classic latent variable.

The registration problem

- Growing children and weather generate events whose clock times do not coincide.
- This is *phase variation*.
- We have to separate phase variation from intensity or *amplitude variation* before we study either type.

- We start with clock times t_{ij} for events $i = 1, ..., I$ and systems $j = 1, ..., N$, but we need to transform to system times
$$\tau_{ij} = h_j(t_{ij})$$
so that the system times for event i are identical.
- The clock–system time transformation $h_j(t)$ for system j is usually smooth and strictly monotonic.
- Transformation $h_j(t)$ is often called a *time warping function*.

$h(t)$ as a positive growth process

- The change in the growth rate $Dh(t)$ can be expressed as

  $$Dh(t_{i+1}) - Dh(t_i) = w(t_i)Dh(t_i)(t_{i+1} - t_i)$$

- This says that the growth rate $Dh(t)$ is proportional to
  - the change in time $(t_{i+1} - t_i)$
  - the growth rate itself
  - a time-varying modulation function $w(t)$

- Rearrangement and taking the limit $\Delta t \to 0$ yields the fundamental differential equation of positive growth:

$$D^2 h(t) = w(t) Dh(t)$$

- The growth differential equation is linear, and has the solution

$$h(t) = \beta_0 + \beta_1 \int_0^t \exp[\int_0^u w(v) dv] du$$

where β_0 is arbitrary and $\beta_1 > 0$.

- the modulation function $w(t)$ is the *functional parameter* of growth, and
  - $w(t) = 0$ implies linear growth
  - $w(t) > 0$ implies increasing growth
  - $w(t) < 0$ implies decreasing growth

- Clock time is at the origin of this functional coordinate system.

A basis expansion for $w(t)$

- Since $w(t)$ is unconstrained, it is natural to represent it as

$$w(t) = \sum_{k}^{K} c_k \phi_k(t)$$

where $\phi_k(t)$ are K basis functions. A B-spline basis is usually appropriate.

- We can also use an expansion of

$$W(t) = \int_0^t w(u) du$$

provided that $W(0) = 0$. For a B-spline basis, this means $c_k = 0$.

Landmark registration

- Given
  - clock times t_{ij} for I identifiable landmarks for unit j and
  - clock times t_{i0} for a template unit,

  we can estimate $h_j(t)$ by monotone smoothing of the relationship between t_{ij} and t_{i0}.
- The registered function $x_j(\tau) = x_j[h_j(t)]$ is computed by estimating the relationship between $x_j(t)$ and $h_j^{-1}(t)$.
- A good initial estimate of inversion function $h_j^{-1}(t)$ is defined by modulation function $-w_j(t)$.
- Events for $x_j(\tau)$ are now synchronized, and exhibit pure amplitude variation.

The template function $x_0(t)$

- This can be iteratively estimated by beginning with

$$x_0(t) = \bar{x}(t) = N^{-1} \sum_j^N x_j(t),$$

registering to this template, and then updating the template using the registered functions. One iteration is usually enough.

- For long time series, we can smooth with a Fourier basis with period equal to one season. This can also be updated.

Continuous registration

- Events are not always easily identified, or we may want to further tune our registration after a preliminary landmark registration.
- We choose a continuous criterion $F(h_j)$ that assesses the misalignment of curve features.
- My personal favorite for F is the smallest eigenvalue of

$$\begin{bmatrix} \int x_0^2(t)\,dt & \int x_0(t)x[h(t)]\,dt \\ \int x_0(t)x[h(t)]\,dt & \int x[h(t)]x[h(t)]\,dt \end{bmatrix}$$

- F close to zero implies that curve values plot against those for the template as a straight line.

Statistics for stochastic time

- For a sample of N units, we can summarize the variation in system times by the variation in the functions $w_j(t)$
- The coefficients c_{jk} offer a finite dimensional summary of the $w_j(t)$'s.
- We can use functional principal components analysis to see their main components of variation.

Montreal daily temperature data

- 34 years of daily temperatures, 1961-1994 inclusive
- Values are averages of daily maximum and minimum
- 12410 observations in tenths of a degree Celsius
- Available for Montreal and 34 other Canadian weather stations

Smoothing the temperature data

The registration process requires that we smooth the data in two ways:

- With an unconstrained smooth that removes the day-to-day variation, but leaves longer-term variation intact.
- With a strictly periodic smooth that eliminates all but strictly seasonal trend.

Unconstrained smooth

- A B-spline expansion usng 500 basis functions of order 6
- Knot about every 25 days
- The standard deviation estimate for residuals about this smooth, adjusted for degrees of freedom, is 4.30 deg C.

Periodic smooth

- 9 Fourier basis functions, period one year
- Knot about every 25 days
- The standard deviation estimate for residuals is 4.74 deg C.

Variance Components

- The std. dev. of the difference between B-spline and Fourier smooths is 2.15 C.
- The std. dev. of the difference between the registered and periodic smooths is 1.73 C.
- $(2.15^2 - 1.73^2)/2.15^2 = 0.35$, the proportion of the smooth variation due to phase.
- The std. dev. of the difference between the raw data and the registered smooths is 4.52 C., as compared to 4.30 for the unregistered smooth.
- 9.5% of the total variation is due to phase.